THREE-DIMENSIONAL LINEARIZED STEADY-STATE PROBLEMS FOR MICROPOLAR VISCOUS LIQUID MEDIA

M. D. Martynenko and Murad Dimian

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Linearized steady-state boundary-value problems are posed within the micropolar (asymmetric) theory of liquid media, and analysis of their uniqueness is carried out.

The equations of motion for micropolar liquid media contain eight unknown quantities, namely, three components of the linear velocity v, three components of the angular velocity Ω, pressure p, and density ρ [1]. For linear steady-state flows of homogeneous viscous liquid media, the number of these equations reduces to seven and they have the following form [1]:

$$(\mu + \alpha)\,\Delta \mathbf{v} + (\mu + \lambda - \alpha)\,\mathrm{grad\,div}\,\mathbf{v} + 2\alpha\,\mathrm{rot}\,\mathbf{\Omega} - \mathrm{grad}\,p = \rho \mathbf{f}\,,$$

$$(\nu + \beta)\,\Delta \mathbf{\Omega} + (\varepsilon + \nu - \beta)\,\mathrm{grad\,div}\,\mathbf{\Omega} + 2\alpha\,\mathrm{rot}\,\mathbf{v} - 4\alpha\,\mathbf{\Omega} = \rho \mathbf{m}\,,$$

$$-\,\mathrm{div}\,\mathbf{v} = 0\,, \tag{1}$$

or in matrix form

$$A\left(\frac{\partial}{\partial x}\right) V = F\,, \tag{2}$$

where

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ p \end{pmatrix}, \quad F = \rho \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ m_1 \\ m_2 \\ m_3 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} (\mu + \alpha)\,\Delta & 0 & 0 \\ 0 & (\mu + \alpha)\,\Delta & 0 \\ 0 & 0 & (\mu + \alpha)\,\Delta \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 0 & -2\alpha\partial_3 & 2\alpha\partial_2 & -\partial_1 \\ 2\alpha\partial_3 & 0 & -2\alpha\partial_1 & -\partial_2 \\ -2\alpha\partial_2 & 2\alpha\partial_1 & 0 & -\partial_3 \end{pmatrix},$$

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$$A_3 = \begin{pmatrix} 0 & -2\alpha\partial_3 & 2\alpha\partial_2 \\ 2\alpha\partial_3 & 0 & -2\alpha\partial_1 \\ -2\alpha\partial_2 & 2\alpha\partial_1 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} (\nu+\beta)\Delta + 2\kappa\partial_1^2 - 4\alpha & 2\kappa\partial_1\partial_2 & 2\kappa\partial_1\partial_3 & 0 \\ 2\kappa\partial_1\partial_2 & (\nu+\beta)\Delta + 2\kappa\partial_2^2 - 4\alpha & 2\kappa\partial_2\partial_3 & 0 \\ 2\kappa\partial_1\partial_3 & 2\kappa\partial_2\partial_3 & (\nu+\beta)\Delta + 2\kappa\partial_3^2 - 4\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following notation is used here:

$$\partial_i = \frac{\partial}{\partial x_i}\,(i = \overline{1,\,3}\,), \quad \kappa = \frac{\varepsilon + \nu - \beta}{2}, \quad \alpha = -\gamma,$$

Here λ, μ, γ, θ, τ, β, and η are material constants of the medium, satisfying the inequalities [1]

$$\mu > 0\,, \ 3\lambda + 2\mu > 0\,, \ \mu - \gamma > 0\,, \ -\gamma > 0\,, \ \theta + \tau > 0\,, \ \theta - \tau > 0\,, \ \rho > 0\,.$$

For α, ε, ν, and β we have the following limitations:

$$\alpha > 0\,, \ \varepsilon > 0\,, \ \nu > 0\,, \ \beta > 0 \ \ (2\theta = \nu + \beta\,; \ 2\eta = \nu - \beta\,; \ \varepsilon = 2\tau\,; \ \gamma = -\alpha)\,.$$

For system of equations (1) it is possible to obtain formulas similar to the well-known Green's formula in the theory of harmonic functions. For this, U will denote the column formed by the components of the vectors **u**, $\boldsymbol{\omega}$, and q, so that $U' = (u_1, u_2, u_3, \omega_1, \omega_2, \omega_3, q)$. Then

$$U' A\left(\frac{\partial}{\partial x}\right) V = (\mu + \alpha)\,(\mathbf{u}\,,\ \Delta \mathrm{v}) + (\mu + \lambda - \alpha)\,(\mathbf{u}\,,\ \operatorname{grad}\operatorname{div}\mathrm{v}) +$$

$$+ 2\alpha\,(\mathbf{u}\,,\ \operatorname{rot}\boldsymbol{\Omega}) - (\mathbf{u}\,,\ \operatorname{grad} p) + (\nu+\beta)\,(\boldsymbol{\omega}\,,\ \Delta\boldsymbol{\Omega}) + (\varepsilon + \nu - \beta)\,\times$$

$$\times\,(\boldsymbol{\omega}\,,\ \operatorname{grad}\operatorname{div}\boldsymbol{\Omega}) + 2\alpha\,(\boldsymbol{\omega}\,,\ \operatorname{rot}\mathrm{v}) - 4\alpha\,(\boldsymbol{\omega}\,,\ \boldsymbol{\Omega}) - q\operatorname{div}\mathrm{v}\,,$$

whence

$$U' A\left(\frac{\partial}{\partial x}\right) V + E(U,\ V) = \sum_{j=1}^{3} \partial_j R_j(U,\ V)\,, \tag{3}$$

where

$$E(U,\ V) = \frac{3\lambda + 2\mu}{3}\operatorname{div}\mathbf{u}\operatorname{div}\mathrm{v} + \frac{3\varepsilon + 2\nu}{3}\operatorname{div}\boldsymbol{\Omega}\operatorname{div}\boldsymbol{\omega} - p\operatorname{div}\mathrm{v} - q\operatorname{div}\mathbf{u} +$$

$$+ \frac{\mu}{2}\sum_{i,j=1}^{3}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\operatorname{div}\mathbf{u}\right]\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\operatorname{div}\mathrm{v}\right] +$$

$$+ \frac{\alpha}{2}\sum_{i,j=1}^{3}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + 2\sum_{k=1}^{3}\varepsilon_{ijk}\,\omega_k\right]\left[\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} + 2\sum_{k=1}^{3}\varepsilon_{ijk}\,\Omega_k\right] +$$

$$+\frac{\nu}{2}\sum_{i,j=1}^{3}\left[\frac{\partial\Omega_i}{\partial x_j}+\frac{\partial\Omega_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\operatorname{div}\boldsymbol{\Omega}\right]\left[\frac{\partial\omega_i}{\partial x_j}+\frac{\partial\omega_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\operatorname{div}\boldsymbol{\omega}\right]+$$

$$+\frac{\beta}{2}\sum_{i,j=1}^{3}\left[\frac{\partial\Omega_i}{\partial x_j}-\frac{\partial\Omega_j}{\partial x_i}\right]\left[\frac{\partial\omega_i}{\partial x_j}-\frac{\partial\omega_j}{\partial x_i}\right], \quad (4)$$

$$R_j(U, V)=\sum_{i=1}^{3}[u_i\sigma_{ij}+\omega_i\mu_{ij}], \quad (5)$$

$$\sigma_{ij}=(-p+\lambda\operatorname{div}\mathbf{v})\,\delta_{ij}+(\mu+\alpha)\frac{\partial v_i}{\partial x_j}+(\mu-\alpha)\frac{\partial v_j}{\partial x_i}+2\alpha\sum_{k=1}^{3}\varepsilon_{ijk}\Omega_k,$$

$$\mu_{ij}=\varepsilon\delta_{ij}\operatorname{div}\boldsymbol{\Omega}+(\nu+\beta)\frac{\partial\Omega_i}{\partial x_j}+(\nu-\beta)\frac{\partial\Omega_j}{\partial x_i}. \quad (6)$$

Therefore, after integration over the finite region D with a boundary S of the Lyapunov type and the external normal $\mathbf{n}$, we obtain

$$\int_D\left[U' A\left(\frac{\partial}{\partial x}\right)V+E(U, V)\right]dx=\int_S\sum_{i,j=1}^{3}[u_i\sigma_{ij}+\omega_i\mu_{ij}]\,n_j\,dS. \quad (7)$$

If D is an infinite region (the exterior of the surface S), then for formula (7) to be valid it is necessary that it be regular at infinity. A more accurate analysis shows that the components of the linear velocity u, v and ω must decrease as $1/|x|$ and their first-order derivatives as $1/|x|^2$ as $|x|\to\infty$; p and q must decrease as $1/|x|^2$ as $|x|\to\infty$.

If ($\mathbf{v}$, $\boldsymbol{\Omega}$, p) is a solution of system (1), all the above formulas are simplified due to the third equation in this system.

For system (1) (or (2), which is the same), unique problems are those with boundary conditions of one of the following types:

type I:

$$\mathbf{v}|_S=\mathbf{a}, \quad \boldsymbol{\Omega}|_S=\mathbf{b}, \quad (8)$$

type II:

$$-pn_i+\sum_{j=1}^{3}\left[(\mu+\alpha)\frac{\partial v_i}{\partial x_j}+(\mu-\alpha)\frac{\partial v_j}{\partial x_i}+2\alpha\sum_{k=1}^{3}\varepsilon_{ijk}\Omega_k\right]n_j|_S=C_i,$$

$$\varepsilon n_i\operatorname{div}\boldsymbol{\Omega}+\sum_{j=1}^{3}\left[(\nu+\beta)\frac{\partial\Omega_i}{\partial x_j}+(\nu-\beta)\frac{\partial\Omega_j}{\partial x_i}\right]n_j|_S=d_i, \quad i=\overline{1,3}. \quad (9)$$

The above formulas can be used to test these boundary-value problems for uniqueness in the following way. Assuming that for the corresponding problem two solutions exist, we obtain for their difference a homogeneous boundary-value problem ($\mathbf{f}=\mathbf{m}=0$, $\mathbf{a}=\mathbf{b}=0$ for the first type of boundary conditions and $C_i=d_i=0$, $i=\overline{1,3}$ for the second type of boundary conditions). For this difference $V_0=V_1-V_2$ formula (7) gives

$$\int_D E_0(V_0, V_0)\,dx=0, \quad (10)$$

where

$$E_0(V_0, V_0) = \frac{3\varepsilon + 2\nu}{3}(\operatorname{div} \mathbf{\Omega}_0)^2 + \frac{\mu}{2} \sum_{i,j=1}^{3} \left[\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{0j}}{\partial x_i} \right]^2 +$$

$$+ \frac{\alpha}{2} \sum_{i,j=1}^{3} \left[\frac{\partial v_{0i}}{\partial x_j} - \frac{\partial v_{0j}}{\partial x_i} + 2 \sum_{k=1}^{3} \varepsilon_{ijk} \Omega_{0k} \right]^2 +$$

$$+ \frac{\nu}{2} \sum_{i,j=1}^{3} \left[\frac{\partial \Omega_{0i}}{\partial x_j} + \frac{\partial \Omega_{0j}}{\partial x_i} - \frac{2}{3} \delta_{ij} \operatorname{div} \Omega_0 \right]^2 + \frac{\beta}{2} \sum_{i,j=1}^{3} \left[\frac{\partial \Omega_{0i}}{\partial x_j} - \frac{\partial \Omega_{0j}}{\partial x_i} \right]^2 . \tag{11}$$

It follows from the positive definiteness of $E_0(V_0, V_0)$ that equality (9) is only possible for $E_0(V_0, V_0) \equiv 0$, i.e., when the following equalities are identically fulfilled in D:

$$\operatorname{div} \mathbf{\Omega}_0 = 0\,, \quad \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{0j}}{\partial x_i} = 0\,, \quad \frac{\partial v_{0i}}{\partial x_j} - \frac{\partial v_{0j}}{\partial x_i} + 2 \sum_{k=1}^{3} \varepsilon_{ijk} \Omega_{0k} = 0\,,$$

$$\frac{\partial \Omega_{0i}}{\partial x_j} + \frac{\partial \Omega_{0j}}{\partial x_i} - \frac{2}{3} \delta_{ij} \operatorname{div} \mathbf{\Omega}_0 = 0\,, \quad \frac{\partial \Omega_{0i}}{\partial x_j} - \frac{\partial \Omega_{0j}}{\partial x_i} = 0\,, \quad i, j = \overline{1, 3}\,,$$

whence

$$\frac{\partial v_{0i}}{\partial x_j} + \sum_{k=1}^{3} \varepsilon_{ijk} \Omega_{0k} = 0\,, \quad \frac{\partial \Omega_{0i}}{\partial x_j} = 0\,, \quad i, j = 1, 2, 3\,,$$

and therefore

$$\mathbf{\Omega}_0 = \mathbf{a}_0\,, \quad \mathbf{v}_0 = \mathbf{a}_0 \times \mathbf{x} + \mathbf{b}_0\,, \tag{12}$$

where $\mathbf{a}_0$ and $\mathbf{b}_0$ are arbitrary real constant vectors in E_3 that can be determined only from the boundary conditions on S and the conditions at infinity (in the case of external problems). Because of this it can be stated that the internal problem with the first type of boundary conditions and the external problems with the first and second types of boundary conditions admit at least one solution (regular at infinity for the external problems). Any two solutions of the internal problem with the second type of boundary conditions can differ by an expression of type (12).

For system (1) other types of boundary problems can be formulated when the boundary conditions are combinations of conditions of types I and II. Their test for uniqueness is carried out following the same procedure.

As follows from the form of system (1), the pressure p is determined within an additive constant. For external boundary problems this constant is zero because the desired pressure must tend to zero at infinity. In the case of the internal boundary problem with boundary conditions of the first-type it remains undetermined.

The third equation of system (1) yields the following condition immediately:

$$\int_S (\mathbf{v}, \mathbf{n})\, dS = 0\,, \tag{13}$$

and therefore in the case of the internal problem with a boundary condition of the first type, a necessary condition for its solvability has the form

$$\int_S (\mathbf{a}, \mathbf{n})\, dS = 0\,.$$

The same condition fixes the arbitrariness in the choice of the vectors $\mathbf{a}_0$ and $\mathbf{b}_0$ in formula (12).

The present study should be considered an extension of the classical results of [2] on the uniqueness of the boundary-value problems for the system of Navier-Stokes equations in the dynamics of viscous incompressible liquids with a symmetrical stress tensor. The solvability of boundary-value problems for system (1) can be tested by functional methods or by construction of a suitable theory of hydrodynamic potentials.

NOTATION

v, linear velocity vector, $\mathbf{v} = (v_1, v_2, v_3)$; Ω, angular velocity vector, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$; p, pressure; ρ, density; $\mathbf{f} = (f_1, f_2, f_3)$, volume-distributed forces; $\mathbf{m} = (m_1, m_2, m_3)$, volume-distributed moments; λ and μ, volumetric and shear viscosities; η, θ, τ, rotational viscosities; γ, measure of cohesion of a liquid particle with its environment; $\alpha = -\gamma$, $\varepsilon = 2\eta$, $\nu = \theta + \tau$, $\beta = \theta - \tau$, $\kappa = \tau + \eta$; $\partial_i = \partial/\partial x_i$, $i = 1, 2, 3$; **x**, position vector of an instantaneous point in the Euclidean space E_3; (**a**, **b**), scalar product of the vectors **a** and **b** in E_3; $\mathbf{a}\times\mathbf{b}$, vector product of the vectors **a** and **b** in E_3; δ_{ij}, Kronecker delta: $\delta_{ii} = 1$; $\delta_{ij} = 0$, $i \neq j$; ε_{ijk}, Levi-Civita symbol: $\varepsilon_{ijk} = +1$ or $\varepsilon_{ijk} = -1$ if i, j, k form an add or even permutation of the numbers 1, 2, 3, $\varepsilon_{ijk} = 0$ if $i = j$ or $i = k$ or $j = k$; Δ, Laplacian; σ_{ij}, stress tensor components; μ_{ij}, micromoment tensor components; D, region in ε_3; S, its boundary; **n**, vector of the unit normal to S; $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

REFERENCES

1. É. L. Aéro, A. N. Bulygin, and E. V. Kuvshinskii, Prikl. Mat. Mekh., **29**, No. 2, 297-308 (1964).
2. O. A. Ladyzhenskaya, Mathematical Problems of Inviscid Fluid Dynamics [in Russian], Moscow (1970), p. 288.
3. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity [in Russian], Moscow (1976), p. 664.